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Lie Algebras and Quasifield Operators

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Abstract

It is shown that realisations of any Lie algebra by means of bilinear polynomials of quasifield operators exist. These realisations are used to find some class of representations of the algebra.

1. *Introduction*

In the present paper we show that any Lie algebra $\mathscr{A}$ can be realised in terms of polynomials of operators which satisfy partially Bose and partially Fermi commutation relations.

The realisations of some Lie algebras by means of Bose operators a_i, b_j, $i, j = 1,\ldots, n$, $[a_i,a_j] = 0$, $[b_i,b_j] = 0$, $[a_i,b_j] = \delta_{ij}$ was known long ago and during the last years has been often used for physical applications. A common method for embedding of an arbitrary Lie algebra $\mathscr{A}$ in the set of bilinear combinations $b_i a_j$ of the Bose operators has been worked out by Palev (1968). Doebner & Palev (1970) have shown that the elements of $\mathscr{A}$ can be expressed as rational functions of a_i and b_j, $i, j = 1,\ldots, n$. Another generalisation of the embedding method was recently given by Kademova (1969). There, it has been shown that some of the realisations obtained by Doebner & Palev (1970)—those which are bilinear with respect to a_i and b_j are valid for a much larger class of operators—the so-called para-Bose and para-Fermi operators.

The aim of our paper is to combine the results from Palev (1968) and Kademova (1970) with some old results obtained by Green (1953). In this way we enlarge the space where the representations of the algebra are realised, and thus a larger class of representations can be obtained. An advantage of the present approach is the fact that the basis of the representation space is known. Therefore it is possible to find all the representations in a matrix form and also to prove the isomorphism between the Lie algebra $\mathscr{A}$ and its realisation with quasifield operators.

The algebra $\mathscr{U}(n,p,\epsilon)$ of the quasifield operators is introduced in Section 2. In Section 3 we prove that any Lie algebra $\mathscr{A}$ can be embedded in $\mathscr{U}(n,p,\epsilon)$. Some class of the representations is discussed in Section 4.

2. *Definition of the Carring Algebra* $\mathscr{U}(n,p,\epsilon)$

Let $\mathbf{F}$ be an arbitrary field and a_i^α and b_j^β, $i, j = 1, \ldots, n$, $\alpha, \beta = 1, \ldots, p$, be given entities. By $T(n,p)$ we denote the algebra of the polynomials of a_i^α and b_j^β over the field $\mathbf{F}$. Let $\mathscr{T}_\epsilon$ be the ideal in $T(n,p)$ generated from the elements:

$$\begin{aligned} &a_i^\alpha b_j^\beta + (-1)^{\delta_{\alpha\beta}} \epsilon b_j^\beta a_i^\alpha - I\delta_{ij}\delta_{\alpha\beta} \\ &a_i^\alpha a_j^\beta + (-1)^{\delta_{\alpha\beta}} \epsilon a_j^\beta a_i^\alpha \\ &b_i^\alpha b_j^\beta + (-1)^{\delta_{\alpha\beta}} \epsilon b_j^\beta a_i^\alpha \end{aligned} \tag{2.1}$$

where α, $\beta = 1, \ldots, p$, $i, j = 1, \ldots, n$, $\epsilon = \pm 1$ (when ϵ is lower-case index, $\epsilon = \pm$), I is the unity of $T(n,p)$.

In what follows we shall compare always the elements in $T(n,p)$ modulo $\mathscr{T}_\epsilon$, i.e. we shall consider the factor algebra

$$\mathscr{U}(n,p,\epsilon) = T(n,p)/\mathscr{T}_\epsilon \tag{2.2}$$

For convenience of the terminology we shall refer to $\mathscr{U}(n,p,\epsilon)$ as quasi-enveloping algebra.

From (2.1) it follows that in $\mathscr{U}(n,p,\epsilon)$ the following relations hold:

$$\begin{aligned} &[a_i^\alpha, b_j^\alpha]_{-\epsilon} = I\delta_{ij}, \qquad [a_i^\alpha, a_j^\alpha]_{-\epsilon} = [b_i^\alpha, b_j^\alpha]_{-\epsilon} = 0 \qquad \text{for } \beta \\ &[a_i^\alpha, b_j^\beta]_\epsilon = [a_i^\alpha, a_j^\beta]_\epsilon = [b_i^\alpha, b_j^\beta]_\epsilon = 0 \qquad \text{for } \alpha \neq \beta \end{aligned} \tag{2.3}$$

Clearly, the set of a_i^α, b_j^α, $i, j = 1, \ldots, n$, for positive ϵ satisfy the commutation relations for Bose operators, whereas if $\alpha \neq \beta$ all the operators anticommute. Therefore we call them quasi-Bose operators. Analogously for negative ϵ the entities a_i^α and b_j^α have the properties of Fermi operators. If $\alpha \neq \beta$ they commute. We can call them quasi-Fermi operators. For convenience of the references by quasifield operators we understand either quasi-Bose or quasi-Fermi operators. With respect to the algebra $\mathscr{U}(n,p,\epsilon)$ the quasifield operators will be referred to as its generators.†

It follows from the structure relations (2.3) that any element $a \in \mathscr{U}(n,p,\epsilon)$ can be expressed as a linear combination of the monomials

$$\prod_{\alpha=1}^{p} \prod_{i=1}^{n} (b_i^\alpha)^{m_i^\alpha} \prod_{\beta=1}^{p} \prod_{j=1}^{n} (a_j^\beta)^{n_j^\beta} \tag{2.4}$$

where m_i^α, n_j^β are positive integers.

† The operators a_i^α, b_j^β have been introduced by Green (1953). Certain combinations of them, namely

$$a_i = \sum_{\alpha=1}^{p} a_i^\alpha \qquad \text{and} \qquad b_j = \sum_{\alpha=1}^{p} b_j^\alpha$$

are called Green Ansatz and satisfy the relations for para-Bose and para-Fermi operators.

The set $\mathscr{B}_\epsilon$ of all elements of the form (2.4) is linearly independent and generates $\mathscr{U}(n,p,\epsilon)$. Therefore, $\mathscr{B}_\epsilon$ forms a basis in the space of the algebra $\mathscr{U}(n,p,\epsilon)$.

3. *Embedding of an Arbitrary Lie Algebra in $\mathscr{U}(n,p,\epsilon)$*

In the present section we prove that any Lie algebra $\mathscr{A}$ is contained in a properly chosen quasi-enveloping algebra. The exact statement is the following:

Theorem 1

For an arbitrary finite Lie algebra $\mathscr{A}$ there exists a positive integer n such that the algebra $\mathscr{U}(n,p,\epsilon)$ contains a Lie-isomorphic image of $\mathscr{A}$ for any p and ϵ.†

The proof of this Theorem can be derived from the properties of the Green Ansatz and the Proposition 1 given by Kademova (1970). In order not to introduce new notations, and to make the exposition selfconsistent, we give a straightforward proof.

Proof: Since any Lie algebra $\mathscr{A}$ has at least one exact finite representation (Ado–Iwasawa theorem, Jacobson, 1967) it is enough to prove the Theorem for an arbitrary algebra of finite matrices.

Consider $\mathscr{A}$ to be an algebra of $n \otimes n$ matrices. To every element $M \in \mathscr{A}$ we put in correspondence an element $M\theta \in \mathscr{U}(n,p,\epsilon)$ which we define to be‡

$$M\theta = \tfrac{1}{2}M_{ij}\left[\sum_{\alpha=1}^{p} b_i^{\alpha}, \sum_{\beta=1}^{p} a_j^{\beta}\right]_\epsilon \tag{3.1}$$

In order to prove the Theorem we have to show that θ is one-to-one mapping, which preserves the commutation relations, i.e. $[M\theta, N\theta] = [M,N]\theta$.

Consider the commutator $[M\theta, N\theta] \equiv F \in \mathscr{U}(n,p,\epsilon)$

$$\begin{aligned} F &= \left[\tfrac{1}{2}M_{ij}\left[\sum_{\alpha=1}^{p} b_i^{\alpha}, \sum_{\beta=1}^{p} a_j^{\beta}\right]_\epsilon, \tfrac{1}{2}N_{lm}\left[\sum_{\gamma=1}^{p} b_l^{\gamma}, \sum_{\delta=1}^{p} a_m^{\delta}\right]_\epsilon\right] \\ &= \tfrac{1}{4}M_{ij}N_{lm}\sum_{\alpha=1}^{p}\,[[b_i^{\alpha}, a_j^{\alpha}]_\epsilon, [b_l^{\alpha}, a_m^{\alpha}]_\epsilon] \end{aligned} \tag{3.2}$$

The last equality in (3.2) follows from the relations (2.3). The fact that any bilinear combination of commuting or anticommuting objects commute has also been used. Using now the identity

$$[ab, c]_- = a[b,c]_{-\epsilon} + \epsilon[a,c]_{-\epsilon}b \tag{3.3}$$

† It is to be understood that the Lie-commutator in the associative algebra $\mathscr{U}(n,p,\epsilon)$ is introduced in a natural way, i.e. if $a, b \in \mathscr{U}(n,p,\epsilon)$ then $[a,b] = ab - ba$.

‡ We adopt the summation convention over repeated low indices.

we obtain:

$$F = \tfrac{1}{4} M_{ij} N_{lm} \sum_{\alpha=1}^{p} (2\delta_{jl}[b_i^{\alpha}, a_m^{\alpha}]_\epsilon - 2\delta_{im}[b_l^{\alpha}, a_j^{\alpha}]_\epsilon)$$
$$= \tfrac{1}{2}[M, N]_{im} \left[\sum_{\alpha=1}^{p} b_i^{\alpha}, \sum_{\beta=1}^{p} a_m^{\beta} \right]_\epsilon = [M, N]\,\theta \tag{3.4}$$

We shall complete the proof by showing that θ is one-to-one. For this we express $M\theta$ as a linear combination of the basis elements (2.4). We have:

$$M\theta = -pT_z\, M.I + \sum_{\alpha=1}^{p} M_{ij} b_i^{\alpha} a_j^{\alpha} \tag{3.5}$$

Clearly, if $M\theta = N\theta$ then $M_{ij} = N_{ij}$ and θ is one-to-one mapping of $\mathscr{A}$ into $\mathscr{A}\theta C\mathscr{U}(n,p,\epsilon)$. We want to point out that everywhere in the proof p and ϵ were arbitrary. ■

The previous Theorem shows that any finite Lie algebra $\mathscr{A}$ can be embedded in a proper chosen algebra $\mathscr{U}(n,p,\epsilon)$. More precisely we can say that $\mathscr{A}$ has a realisation as bilinear combinations of quasifield operators. A natural question arises as to whether this realisation is unique or it is possible to map the algebra $\mathscr{A}$ in different ways in $\mathscr{U}(n,p,\epsilon)$. In particular it is interesting to see whether $\mathscr{A}$ can be realised in terms of higher-order polynomials, and to find all such realisations. In the present paper we shall answer partially this question. We shall show that the embedding is not unique and that there exist realisations which are not bilinear with respect to b_i^{α}, a_j^{β}. The problem of higher-order realisations is closely related to the concept of the canonical isomorphisms of $\mathscr{U}(n,p,\epsilon)$ (Doebner & Palev, 1970). A canonical isomorphism of $\mathscr{U}(n,p,\epsilon)$ is a mapping $\tilde{c}$ of the quasifield operators $a_i^{\alpha}, b_j^{\beta}, i,j = 1,\ldots,n, \alpha,\beta = 1,\ldots,p$, into $\mathscr{U}(n,p,\epsilon)$ which preserves the structure relations (2.3). Suppose that there exists a canonical isomorphism $\tilde{c}$ such that the images of the quasifield operators $\tilde{a}_i^{\alpha}$, $\tilde{b}_j^{\beta}$ are polynomials in a_i^{α}, b_j^{β}. Since in the proof of Theorem 1 only the structure relations were used, it follows that the substitution $a_i^{\alpha} \to \tilde{a}_i^{\alpha}$ and $b_j^{\beta} \to \tilde{b}_j^{\beta}$ in the realisation (3.1) of $\mathscr{A}$ gives a new realisation. If $\tilde{a}_i^{\alpha}$ and $\tilde{b}_j^{\beta}$ are higher-order polynomials of the quasifield operators, so is the new realisation of the algebra.

We now proceed to construct one possible kind of canonical isomorphism. Since we intend to give only an example, we shall restrict our considerations to the case of the algebra $\mathscr{U}(n,p,+)$. Let $a_i^{\alpha}, b_j^{\beta}, i,j = 1,\ldots,n, \alpha,\beta = 1,\ldots,p$, be the generators of $\mathscr{U}(n,p,+)$. Determine the mapping $\tilde{c}$ of the generators into $\mathscr{U}(n,p,+)$ in the following way:

$$\begin{aligned} a_i^{\alpha}\,\tilde{c} &= a_i^{\alpha} + \delta_{ik}\,\delta_{\alpha\gamma}(b_k^{\gamma})^{2m-1} \\ b_j^{\beta}\,\tilde{c} &= b_j^{\beta} \end{aligned} \tag{3.6}$$

where m, k and γ are fixed positive integers such that $1 \leqslant k \leqslant n$, $1 \leqslant \alpha \leqslant p$, $i,j = 1,\ldots,n$, $\alpha,\beta = 1,\ldots,p$.

A simple verification shows that the operators $a_i^\alpha \tilde{c}$ and $b_j^\beta \tilde{c}$ satisfy the structure relations (2.3) and hence $\tilde{c}$ is a canonical isomorphism of $\mathscr{U}(n,p,+)$. Moreover, the operators $a_i^\alpha \tilde{c}$ and $b_j^\beta \tilde{c}$ are polynomials of order $2m-I$ of the generators. Therefore, the substitution $a_i^\alpha \to a_i^\alpha \tilde{c}$ and $b_j^\beta \to b_j^\beta \tilde{c}$ in (3.1) immediately leads to a new realisation of the algebra $\mathscr{A}$ through polynomials of order $2m$ of the quasifield Bose operators. Thus we have established the following:

Proposition 1

Any finite Lie algebra over an arbitrary field can be realised in terms of polynomials of an arbitrary even order of the quasifield operators.

The set Γ of all canonical isomorphisms of the type (3.6) is not closed with respect to multiplication. It can be easily checked, however, that the product of any two elements $\tilde{c}_1 \in \Gamma$ and $\tilde{c}_2 \in \Gamma$ is also a canonical isomorphism. This provides a new possibility for constructing a larger class of canonical isomorphisms. Indeed, denote by $\bar{\Gamma}$ the set of all monomials

$$\prod_{k=1}^{p} \tilde{c}_k^{n_k}$$

of arbitrary elements $\tilde{c}_k \in \Gamma$. Then any element from $\bar{\Gamma}$ is a canonical isomorphism and Γ is a proper subset of $\bar{\Gamma}$.

Concluding the discussion about higher-order realisations we would like to remark that in the quasi-Fermi case there is no need in looking for polynomials of order higher than $2np$, since these are the highest polynomials in $\mathscr{U}(n,p,-)$. In this case it is not possible to construct canonical isomorphisms in a way similar to that considered above [see (3.6)], since $(b_i^\alpha)^n = 0$ for $n > 1$.

However, such realisations exist. We give an example of this kind for the algebra $o(2,2)$.†

Let us specify the generators of the algebra in the following matrix form:

$$L_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad L_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad L_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$L_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad L_5 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad L_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{3.7}$$

† A general approach for constructing second-order realisations of an arbitrary Lie algebra by means of para-Fermi or para-Bose creation and annihilation operators is given in Kademova & Palev (1970).

Consider now the following second-order polynomials of two Fermi operators:

$$\begin{array}{lll} \mathscr{L}_1 = [b_1, a_1]_- & \mathscr{L}_2 = [b_1, a_2]_- & \mathscr{L}_3 = [b_2, a_1]_- \\ \mathscr{L}_4 = [b_2, a_2]_- & \mathscr{L}_5 = [b_1, b_2]_- & \mathscr{L}_6 = [a_1, a_2]_- \end{array} \tag{3.8}$$

One can easily check that with respect to the mapping $L_i \rightarrow \mathscr{L}_i$ the polynomials (3.8) form a realisation of $o(2,2)$.

4. *Representations of an Arbitrary Lie Algebra*

In this section we show how the mapping of an arbitrary Lie algebra $\mathscr{A}$ in $\mathscr{U}(n,p,\epsilon)$ can be used in order to find some class of representations of $\mathscr{A}$.

Let $\bar{\mathscr{A}}$ be the realisation of $\mathscr{A}$ in $\mathscr{U}(n,p,\epsilon)$. We consider the algebra $\mathscr{U}(n,p,\epsilon)$ as a linear space and with each element $a \in \bar{\mathscr{A}}$ we associate a linear transformation a^* of $\mathscr{U}(n,p,\epsilon)$ into itself, which we define in the following way:

$$a^* x = [a, x] \qquad \text{for } \forall x \in \mathscr{U}(n,p,\epsilon) \tag{4.1}$$

Proposition 2

The mapping $\phi: a \rightarrow a^*$ of $\bar{\mathscr{A}}$ onto $\bar{\bar{\mathscr{A}}} = \{a^* | a \in \bar{\mathscr{A}}\}$ preserves the commutation relations. If $\bar{\mathscr{A}}$ does not contain the unity I of $\mathscr{U}(n,p,\epsilon)$ then ϕ is one-to-one mapping. In this case $\bar{\bar{\mathscr{A}}}$ is an exact representation.

Proof: Let a_1 and a_2 be arbitrary elements from $\bar{\mathscr{A}}$ and x an arbitrary element from $\mathscr{U}(n,p,\epsilon)$.

$$\begin{aligned}[] [a_1^*, a_2^*] x = [a_1, [a_2, x]] &= a_1 a_2 x - a_2 a_1 x - x a_1 a_2 + x a_2 a_1 \\ &= [a_1, a_2]^* x \end{aligned}$$

To complete the proof let us remark that the kernel of $\mathscr{U}(n,p,\epsilon)$ is generated from the unity of this algebra.

Suppose now that $a_1^* = a_2^*$. Then for any $x \in \mathscr{U}(n,p,\epsilon)$, $a_1^* x = a_2^* x$, and therefore $[a_1 - a_2, x] = 0$. Hence $a_1 - a_2 = \alpha 1$, where α is an element of the ground field $\mathbf{F}$. If $\bar{\mathscr{A}}$ does not contain the unity, then $\alpha 1 \in \bar{\mathscr{A}}$ only for $\alpha = 0 \in \mathbf{F}$, and hence $a_1 = a_2$. ■

The representations $\bar{\bar{\mathscr{A}}}$ of $\mathscr{A}$ we have constructed above is, however, not very useful if it is necessary for it to be written in a matrix form. The space $\mathscr{U}(n,p,\epsilon)$ is noncommutative, and to express the element $a^* x$ through the basis one has, therefore, to use many times the structure relations (2.3).

It is more convenient to realise the representations of $\mathscr{A}$ in the subspace V of $\mathscr{U}(n,p,\epsilon)$ consisting of all polynomials of the operators b_i^α, $i = 1, \ldots, n$, $\alpha = 1, \ldots, p$. This is not possible always, since for some realisations $\bar{\mathscr{A}}$

of $\mathscr{A}$ the space V is not $\bar{\mathscr{A}}$-space, i.e. it is not closed under the mapping $x \to a^* x$, $a \in \bar{\mathscr{A}}$. To prove that there exist realisations of $\mathscr{A}$ in $\mathscr{U}(n,p,\epsilon)$ for which V can carry representations, we are going to establish first the following:

Proposition 3

The space $V \subset \mathscr{U}(n,p,\epsilon)$ is closed under the mapping $x \to [[b_i^\alpha, a_j^\alpha]_\epsilon, x]$ for any possible values of i, j and α.

Proof: Since the mapping under consideration is a linear one, and the space V is spanned on all vectors,

$$(m_1^1, \ldots, m_n^p) \equiv (b_1')^{m_1'} (b_2')^{m_2'} \cdots = \prod_{\beta=1}^{p} \prod_{k=1}^{n} (b_k^\beta)^{m_k^\beta} \tag{4.2}$$

it is enough to show that

$$[[b_i^\alpha, a_j^\alpha]_\epsilon, (m_1^1, \ldots, m_n^p)] \in V$$

Calculating directly we get

$$[[b_i^\alpha, a_j^\alpha]_\epsilon, (m_1^1, \ldots, m_n^p)] = \prod_{\beta=1}^{\alpha-1} \prod_{k=1}^{n} (b_k^\beta)^{m_k^\beta} \mathscr{D}_{ij}^\alpha \prod_{\gamma=\alpha+1}^{p} \prod_{t=1}^{n} (b_t^\gamma)^{m_t^\gamma}$$

where

$$\mathscr{D}_{ij}^\alpha = [[b_i^\alpha, a_j^\alpha]_\epsilon, \prod_{s=1}^{n} (b_s^\alpha)^{m_s^\alpha}]$$

Using the structure relations (2.3), after some calculations we obtain finally for $\mathscr{D}_{ij}^\alpha$

$$\mathscr{D}_{ij}^\alpha = \begin{cases} m_j^\alpha \, \epsilon^{\sum_{l=j+1}^{i-1} m_l^\alpha} \prod_{r=1}^{j-1} (b_r^\alpha)^{m_r^\alpha} (b_j^\alpha)^{m_j^\alpha - 1} \prod_{q=j+1}^{i-1} (b_q^\alpha)^{m_q^\alpha} (b_i^\alpha)^{m_i^\alpha + 1} \times \\ \quad \times \prod_{s=i+1}^{n} (b_s^\alpha)^{m_s^\alpha} \qquad \text{for } i > j \\ m_j^\alpha \prod_{r=1}^{n} (b_r^\alpha)^{m_r^\alpha} \qquad \text{for } i = j \\ m_j^\alpha \, \epsilon^{\sum_{l=i+1}^{j-1} m_l^\alpha} \prod_{r=1}^{i-1} (b_r^\alpha)^{m_r^\alpha} (b_i^\alpha)^{m_i^\alpha + 1} \prod_{q=i+1}^{j-1} (b_q^\alpha)^{m_q^\alpha} (b_j^\alpha)^{m_j^\alpha - 1} \times \\ \quad \times \prod_{s=j+1}^{n} (b_s^\alpha)^{m_s^\alpha} \qquad \text{for } i < j \end{cases}$$

Then we can write

$$[[b_i^\alpha, a_j^\alpha]_\epsilon, (m_1^1, \ldots, m_n^p)] = m_j^\alpha \times$$

$$\times \begin{cases} \epsilon^{\sum_{l=j+1}^{i-1} m_l^\alpha} \times (m_1^1, \ldots, m_j^\alpha - 1, \ldots, m_i^\alpha + 1, \ldots, m_n^p) & \text{for } i > j \\ (m_1^1, \ldots, m_i^\alpha, \ldots, m_n^p) \quad \text{for } i = j & \\ \epsilon^{\sum_{l=i+1}^{j-1} m_l^\alpha} \times (m_1^1, \ldots, m_i^\alpha + 1, \ldots, m_j^\alpha - 1, \ldots, m_n^p) & \text{for } i < j \blacksquare \end{cases} \tag{4.3}$$

It follows from Theorem 1 that any Lie algebra $\mathscr{A}$ can be isomorphically mapped in a proper $\mathscr{U}(n,p,\epsilon)$ in such a way that the elements of the image algebra $\bar{\mathscr{A}}$ are linear combinations of some $[b_i^\alpha, a_j^\alpha]_\epsilon \in \mathscr{U}(n,p,\epsilon)$. This result, together with Proposition 3, leads immediately to the following:

Theorem 2

Let $\bar{\mathscr{A}}$ be a realisation of an arbitrary Lie algebra $\mathscr{A}$ in terms of linear combinations of the elements $[b_i^\alpha, a_j^\alpha]_\epsilon \in \mathscr{U}(n,p,\epsilon)$ as defined in Theorem 1, and let V be the space spanned on all polynomials of b_i^α for all α and i. To every $a \in \bar{\mathscr{A}}$ we put in correspondence a linear mapping a^* of V in $\mathscr{U}(n,p,\epsilon)$ defined as

$$a^* x = [a, x] \qquad \forall x \in V \tag{4.4}$$

Then, V is closed under the mapping a^* for all $a \in \bar{\mathscr{A}}$ and the set $\bar{\bar{\mathscr{A}}} = \{a^* | a \in \mathscr{A}\}$ is a representation of $\mathscr{A}$ in V.

The space V is not the sole subspace of $\mathscr{U}(n,p,\epsilon)$ which can carry representations of $\mathscr{A}$. It is possible to find as well many other subspaces which are closed under the same realisation $\bar{\mathscr{A}}$ of the algebra $\mathscr{A}$. For instance, the substitution $b_i^\alpha \to a_i^\alpha$ in V for some or all values of α and i leads to a new subspace V' which also is A-subspace. The representations in V and V' are generally different.

To conclude, we want to point out that the results obtained by Palev (1968) appear to be a particular case of the present consideration when $p = 1, \epsilon = +$. We should add, that in a similar way the analytical continuation of the indices labelling the basis vectors $(m_1^1, \ldots, m_n^p)$ of V leads to new representations in the quasi-Bose case. Our generalisation is, however, different from that developed by Doebner & Palev (1970), where rational functions of the creation and annihilation Bose operators are introduced.

Since certain combinations of the quasifield operators satisfy Green structure relations, the present paper extends also the results obtained by Kademova (1970). There exists a natural isomorphism between the realisation in terms of parafield operators and quasifield operators. In our case, however, the representation space is made larger, and hence we obtain more representations.

The representations we have defined in Theorem 2 are ongoing generalisations of the so-called ladder representations often used in physics. For instance, the ladder representations of $\mathscr{U}(n)$ are realised in V of $\epsilon = +$ and $p = 1$. If V' is a space obtained from V through the substitution $b_i \to a_i$, $i = 1, \ldots, q$, then in V' the ladder representations of $u(n-q,q)$ are realised.

Since up to now not all the representations of $\mathscr{U}(p,q)$ are known, it is an interesting problem to investigate the representations for arbitrary p and ϵ, and to see whether they contain some new classes of representations.

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